

## The Study on $(f, d_n)$ Summability of Fourier Series

Binod Kumar Tiwari

Research Scholar,

University Department of Mathematics, Magadh University, Bodhgaya

### Abstract:

Generalizing the result of Kwee (1970) on  $(E, I)$  summability of Fourier series, Chandra (1977) has proved a theorem of  $(E, q)$  summability of Fourier series for  $q > 0$ . Here in the present paper, we have considered

$(f, d_n)$  summability method defined by Smith (1965) and discussed

$(f, d_n)$  summability of Fourier series under very general condition. It may be noted that  $(f, d_n)$  summability method is a generalization of  $(E, q)$  summability.

### 1. Introduction

Let  $f(z)$  be a non constant entire function and let  $\{d_n\}$  be a sequence of complex numbers such that

$$d_i \neq f(0), \quad d_i \neq -f(1) \quad (i \geq 1) \quad (1.1)$$

Smith (1965) defined  $(f, d_n)$  summability method as follows:

An infinite series  $\sum u_n$  with the sequence  $\{S_n\}$  of its partial sums is said to be summable  $(f, d_n)$  to the sum  $S$  if

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} a_{n,k} s_k - S \quad (1.2)$$

where  $a_{n,k}$ 's are the elements of matrix  $A = (a_{n,k})$  defined by the set of equations

$$a_{0,0} = 1$$

$$a_{0,k} = 0 \quad (k \neq 0)$$

and

$$\prod_{i=1}^n \frac{f(z) + d_i}{f(1) + d_i} = \sum_{k=0}^{\infty} a_{n,k} z^k, \quad (n \geq 1)$$

If  $f(z) = z$  and  $d_n = r$ , where  $r$  is any complex constant, we get well known Euler method defined by Agnew (1944). In addition to the assumption made regarding the function  $f$ , we further assume that

$$f(1) = f'(1) = 1$$

While

$$f''(1) \neq 0$$

We write

$$H_n = \sum_{j=1}^n \frac{1}{1+d_j}, L_n = 2 \sum_{j=1}^n \frac{d_j}{(1+d_j)^2}$$

$$V_n = 2H_n, \sigma_n = 4a_n H_n + L_n$$

Where  $a_2$  is constant given by

$$a_2 \frac{f'(1)}{2} > 0 \quad [\text{Shoop (1979)}]$$

Smith (1965) established regularity condition for  $(f, d_n)$  summability method. The regularity condition for  $(f, d_n)$  method imply that

$$v_n \rightarrow \infty, \sigma_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

and  $\frac{v^2 n}{\sigma_n} \rightarrow \infty$

The asymptotic behavior of the Lebesgue constant for  $(f, d_n)$  method of summability of Fourier series has been studied by Shoop (1979). Since the Lebesgue constant is not bounded, it is interesting to study sufficient criteria for  $(f, d_n)$  summability of Fourier series.

Let  $f(t)$  be  $2\pi$ - periodic and Lebesgue integrable function of  $t$  in the interval  $(-\pi, \pi)$ . Then the Fourier series of the function  $f(t)$  is given by

$$f(t) \sim \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \quad (1.4)$$

We write, at a point  $t = x$ ,  $n \rightarrow \infty$

$$\phi(t) = \frac{1}{2} [F(x+1) + f(x-t) - 2f(x)]$$

$$\Phi(t) = \int_0^t |\phi(u)| du$$

## 2. Analysis

Generalizing the result of Kwee (1970) on  $(E, 1)$  summability of Fourier series, Chandra (1977) has studied  $(E, q)$  summability of Fourier series,  $q >$ , by proving the following.

**Theorem A.**

$$\Phi(t) = \int_0^t |\phi(u)| du = O\left(\frac{t}{\log \frac{1}{t}}\right) \quad (2.1)$$

As  $t \rightarrow 0$ , then the Fourier series (1.4) of the function  $F(t)$  is summable  $(E, q)$  to the sum  $F(x)$  at a point  $t = x$  in  $(-\pi, \pi)$

The object of the present paper is to generalize the result of theorem A by considering general summability method  $(f, d_n)$  and very general condition and establishing the following:

**Theorem**

Let  $\alpha(t)$  and  $\beta(t)$  be any two positive functions of  $t$  such that  $\alpha(t)$  and  $\frac{t \alpha(t)}{\beta(t)}$  increase monotonically with  $t$ . Let  $\{P_n\}$  be a non-negative, monotonic non-increasing sequence of constants with  $P_n$  as its non-vanishing  $n$  partial sum tending to infinity as  $n \rightarrow \infty$ .

If

$$\alpha(n) \log n = O[\beta(P_n)] \quad (2.2)$$

as  $n \rightarrow \infty$ , and

$$\Phi(t) = \int_0^t |\phi(u)| du = O\left(\frac{t \alpha \frac{1}{t}}{\beta(P_\tau)}\right) \quad (2.3)$$

as  $n \rightarrow \infty$ , where  $\tau$  is the integral part of then  $\frac{1}{t}$ , then the Fourier series (1.4) of the function  $F(t)$  is summable  $(f, d_n)$  to the sum  $f(x)$  at a point  $t = x$  in  $(-\pi, \pi)$ .

3. The following lemmas are needed in order to prove our theorem

**Lemma 1 : [Shoop (1979)]**

$$\begin{aligned} \prod_{j=1}^n \frac{f(e^{2^n}) + d_j}{1 + d_j} &= e^{(2iH_n t - 4a_2 H_n t^2 - L_n t^2)} + o(H_n t^3) \\ &= e^{(iv_n t - \sigma_n t^2)} + o(v_n t^3) \end{aligned}$$

**Lemma 2 :**

$$\prod_{j=1}^n \frac{f(e^n) + d_j}{1 + d_j} \leq e^{-\frac{a_2 t^2 v_n}{8}}$$

It clearly follows from shoop (1979) on page 260 where

$$\prod_{j=1}^n \frac{R_j}{1 + d_j} \leq e^{-a_2 t^2 H_n}$$

with  $R_j = |f(e^{2^n}) + d_j|$

**4. Proof of the theorem :**

Following Zygmund (1959), if  $T_k(x)$  denotes the  $k^n$  partial sum of the Fourier series (1.4) at a point  $t = x$ , then

$$T_k(x) - F(x) = \frac{2}{\pi} \int_0^\pi \frac{\phi(t)}{t} \sin\left(k + \frac{1}{2}\right)t dt + O(1) \quad (4.1)$$

Therefore,  $(f, d_n)$  transform  $D_n$  of  $T_k$  will be given by following (1.2) as

$$\begin{aligned}
D_n(x)f(x) &= \sum_{k=0}^{\infty} a_{n,k} \{T_k(x) - f(x)\} \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} \left\{ \sum_{k=0}^{\infty} a_{n,k} \sin\left(k + \frac{1}{2}\right)t \right\} dt + O(1) \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} I_n \left\{ e^{\frac{it}{2}} \sum_{k=0}^{\infty} a_{n,k} e^{at} \right\} dt + O(1) \\
&= \frac{2}{\pi} \int_0^{\pi} \frac{\phi(t)}{t} I_n \left\{ e^{\frac{it}{2}} \prod_{j=1}^n \frac{f(e)^{it} + d_j}{1 + d_j} \right\} dt + O(1) \\
&= \left\{ \int_0^{2\pi/v_n} + \int_{2\lambda/v_n}^{v_n-\lambda} + \int_{v_0-\lambda}^{\pi} \right\} + O(1) \\
&= I_1 + I_2 + I_3 + O(1), \text{ say} \tag{4.2}
\end{aligned}$$

Where

$$\frac{1}{3} < \lambda < \frac{1}{2}$$

Let us first consider  $I_1$ . Using lemma 1, we have

$$\begin{aligned}
I_1 &= O(1) \int_0^{2\pi/v_n} \frac{\phi(t)}{t} \left\{ \frac{-\sigma_n t^2}{e^4} \left| \sin(v_n + 1) \frac{t}{2} \right| + O(v_n t^3) \right\} dt \\
&= I_{1,1} + I_{1,2}, \text{ say} \tag{4.3}
\end{aligned}$$

Now,

$$\begin{aligned}
|I_{1,1}| &= O(1) \int_0^{2\pi/v_n} \frac{|\phi(t)|}{t} (v_n + 1) \frac{t}{2} dt \\
&= O(v_n) \int_0^{2\pi/v_n} |\phi(t)| dt \\
&= O(v_n), O\left[ \frac{1 \cdot \alpha(v_n)}{v \beta(Pv_n)} \right], \text{ using (2.3)} \\
&= O\left[ \frac{\alpha(v_n)}{\beta(Pv_n)} \right] \\
&= O\left( \frac{1}{\log v_n} \right) \text{ using (2.2)} \\
&= O(1), \text{ as } \rightarrow \infty \tag{4.4}
\end{aligned}$$

$$\begin{aligned}
|I_{1,2}| &= O(1) \int_0^{2\pi/v_n} \frac{|\phi(t)|}{t} (v_n t^3) dt \\
&= O(v_n) \int_0^{2\pi/v_n} |\phi(t)| \cdot t^2 dt \\
&= O\left(\frac{1}{v_n}\right) \int_0^{2\pi/v_n} |\phi(t)| dt \\
&= O\left(\frac{1}{v_n}\right) O\left[\frac{1}{v_n} - \frac{\alpha(v_n)}{\beta(Pv_n)}\right] \\
&= O\left[\frac{\alpha(v_n)}{v_n^2 \log v_n}\right] \\
O(1), \text{ as } &\rightarrow \infty \tag{4.5}
\end{aligned}$$

From (4.3), (4.4) and (4.5), it follows that

$$I_1 = O(1), \text{ as } n \rightarrow \infty \tag{4.6}$$

Further, considering  $I_2$ , we have

$$I_1 = \frac{2}{\pi} \int_{2\pi/v_n}^{v_n^{-\lambda}} \frac{|\phi(t)|}{t} \left\{ e^{-\sigma_0 t^2} \sin(v_n + t) \frac{t}{2} + O(v_n t^3) \right\} dt \tag{4.47}$$

$$I_{2,1} + I_{2,1} \text{ say} \tag{4.7}$$

$$\begin{aligned}
|I_{2,1}| &= O(1) \int_{2\pi/v_n}^{v_n^{-\lambda}} \frac{|\phi(t)|}{t} \left\{ e^{-\sigma_n t^2} \sin(v_n + r) \frac{t}{2} \right\} dt \\
&= O\left( e^{-\frac{\pi^2 \sigma_n}{v_n^2}} \right) \int_{2\pi/v_n}^{v_n^{-\lambda}} \frac{|\phi(t)|}{t} \left| \sin(v_n + 1) \frac{t}{2} \right| dt \\
&= O \int_{2\pi/v_n}^{v_n^{-\lambda}} \frac{|\phi(t)|}{t} dt \\
&= O(1) \left[ O\left\{ \frac{\alpha\left(\frac{1}{t}\right)}{\beta(P_\tau)} \right\} \right]_{2\pi/v_n}^{v_n^{-\lambda}}
\end{aligned}$$

$$\begin{aligned}
&= O(1) \int_{2\pi/v_n}^{v_n^{-\lambda}} O\left\{\frac{\alpha\left(\frac{1}{t}\right)}{\beta(P_\tau)}\right\} dt \\
&= O\left[\frac{\alpha(v_n^\lambda)}{\beta(v_n^\lambda)}\right] - O\left[\frac{\alpha(v_n)}{\beta(Pv_n)}\right] + O\left[\frac{\alpha(v_n^\lambda)}{\beta Pv_n^\lambda}\right] \int_{2\pi/v_n}^{v_n^{-\lambda}} dt \\
&= O\left(\frac{1}{\log v_n^\lambda}\right) - O\left(\frac{1}{\log v_n}\right) + O\left(\frac{1}{\log v_n^\lambda}\right) \left[\frac{1}{v_n^\lambda} - \frac{1}{v_n}\right] \\
&= O(1), \text{ as } n \rightarrow \infty \tag{4.8}
\end{aligned}$$

and

$$\begin{aligned}
|I_{2.2}| &= O(1) \int_{2\pi/v_n}^{v_n^{-\lambda}} \frac{\phi(t)}{t} (v_n t^2) dt \\
&= O(v_n) \int_{2\pi/v_n}^{v_n^{-\lambda}} |\phi(t)| t^2 dt \\
&= O(v_n^{1-2\lambda}) \int_{2\pi/v_n}^{v_n^{-\lambda}} |\phi(t)| dt \\
&\leq O(v_n^{1-2\lambda}) \int_{2\pi/v_n}^{v_n^{-\lambda}} |\phi(t)| dt \\
&= O(v_n^{1-2\lambda}) O\left[\frac{v_n^{-\lambda} \alpha(v_n^\lambda)}{\beta(Pv_n^\lambda)}\right] \\
&= O(v_n^{1-3\lambda}) O\left(\frac{1}{\log v_n^\lambda}\right) \\
&= O(1), \text{ as } n \rightarrow \infty, \because \frac{1}{3} < \alpha \tag{4.9}
\end{aligned}$$

From (4.7), (4.8) and (4.9), we obtain

$$I_2 = O(1), \text{ as } n \rightarrow \infty, \tag{4.10}$$

Finally, let us consider  $I_3$ . Now using Lemma 2,

$$|I_3| = O(1) \int_{v_n^{-\lambda}}^{\pi} \frac{|\phi(t)|}{t} O\left(e^{-\frac{a_2 v_n t^2}{\delta}}\right) dt$$

$$\begin{aligned}
&= O(v_n^\lambda) O\left(e^{\frac{-u_2 v_n}{8v_n^2 n^\lambda}}\right) \int_{v_n^{-\lambda}}^{\pi} |\phi(t)| dt \\
&\leq O\left(v_n^\lambda \cdot e^{\frac{-ua_2 v_n^{1-2\lambda}}{8}}\right) \\
&= O(1), \text{ as } n \rightarrow \infty \tag{4.11}
\end{aligned}$$

Combining (4.2), (4.6), (4.10) and (4.11), we get the required result. This completes the proof of our theorem.

## 5. References

1. Agnew, R.P. (1944) : Euler transformation. Amer J. Math. Vol. 66, pp. 318-338.
2. Chandra, P. (1977) : Summability of Fourier series by Euler means. Publ. Gras. Goldeano Math. Seminar, Vol. 24 Spanish, pp. 47-52.
3. Kwee, B. (1970) : The Housedorff summability of fourier series. Proc. Amer. Math. Soc. Vol. 24, pp. 586-594.
4. Shoop R.A. (1979) : The Lebesgue constants for (f, d) summability. Pacific J. Math., Vol. 80, No. 10, pp. 235-263.
5. Smith, G. (1965) : On the (f, d) method of summability. Canad. J. Math, Vol. 17, pp. 506-526.
6. Zygmund, A. (1959) : Trigonometrical series volo. I, II<sup>nd</sup> Edition, Cambridge at the University press.

\*\*\*